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$\mathcal{PT}$-Symmetric Klein-Gordon Oscillator

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Abstract Parity-time ($\mathcal{PT}$) symmetric Klein-Gordon oscillator is presented using $\mathcal{PT}$-symmetric minimal substitution. It is shown that wave equation is exactly solvable, and energy spectrum is the same as that of Hermitian Klein-Gordon oscillator presented by Bruce and Minning. Landau problem of $\mathcal{PT}$-symmetric Klein-Gordon oscillator is discussed.

Keywords Klein-Gordon oscillator · $\mathcal{PT}$ symmetry · Landau problem

Although Hermitian Hamiltonian holds mainstream for quantum mechanics and quantum field theory, the study of non-Hermitian Hamiltonian has attracted a great deal of attention. This interest is triggered by the development of the studies of $\mathcal{PT}$-symmetric Hamiltonian. In a fundamental paper [1] Bender and Boettcher found that it is possible to see that the energy eigenvalues of non-Hermitian Hamiltonians such as $H = p^2 + x^2(ix)^\epsilon$ $(\epsilon \geq 0)$ are all real as long as they preserve $\mathcal{PT}$ symmetry. Then, Dorey et al. provided a rigorous proof of spectral positivity [2, 3]. In [4, 5], it was shown that the time-evolution operator for the $\mathcal{PT}$-symmetric Hamiltonian is unitary. A large number of $\mathcal{PT}$-symmetric models have been studied, including $\mathcal{PT}$-symmetric quantum mechanics [6], $\mathcal{PT}$-symmetric quantum electrodynamics [7], $\mathcal{PT}$-symmetric quantum field theory [8] as well as optical $\mathcal{PT}$-symmetric structures [9]. For a comprehensive review of the basic ideas and techniques responsible for the recent developments in non-Hermitian Hamiltonian, see Ref. [10].

Harmonic oscillator is one of the most useful and well studied system. In the non-relativistic limit, the positive energy states of relativistic oscillator reduces to the spectrum of non-relativistic harmonic oscillator. The study of relativistic oscillator is of special interest in particle physics. Following the study of Dirac oscillator first proposed by Moshinsky and Szczepaniak [11], its physical applications and extensions to other case have attracted a lot of attention and been studied intensively by various authors [12–24].

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The Klein-Gordon oscillator was introduced for the first time by Bruce and Minning [14]. The oscillator Hamiltonian becomes quadratic in both the momentum and the coordinates by using a minimal substitution $P \to P - im\hat{\gamma}\hat{\Omega} \cdot Q$. Another alternative definition for Klein-Gordon oscillator was proposed by Mirza and Mohadesi [25], where they showed that the Klein-Gordon and Dirac oscillators in a noncommutative space have a similar behavior to the dynamics of a particle in a commutative space and in a uniform magnetic field. In this paper, we extend Hermitian Klein-Gordon oscillator to $\mathcal{PT}$-symmetric version.

A Hamiltonian is $\mathcal{PT}$-symmetric provided it is invariant under the joint transformation of parity reflection $\mathcal{P}$ and time reversal $\mathcal{T}$. The action of the parity reflection $\mathcal{P}$ and time reversal $\mathcal{T}$ operators is defined as

$$\begin{aligned} &\mathcal{P}: \boldsymbol{x} \to -\boldsymbol{x}, \boldsymbol{p} \to -\boldsymbol{p}, \\ &\mathcal{T}: \boldsymbol{x} \to \boldsymbol{x}, \boldsymbol{p} \to -\boldsymbol{p}, i \to -i. \end{aligned} \tag{1}$$

Let us consider a $\mathcal{PT}$-symmetric minimal substitution

$$\boldsymbol{P} \to \boldsymbol{P} - im\hat{\gamma}\hat{\Omega} \cdot \boldsymbol{Q} - im\hat{\zeta}\hat{\Omega} \cdot \boldsymbol{Q}, \tag{2}$$

where we have employed the same definition for $\boldsymbol{Q}$, $\boldsymbol{P}$, $\hat{\Omega}$, $\hat{\gamma}$ as in [14], namely, $\boldsymbol{Q} = \hat{\eta}\boldsymbol{q}$, $\boldsymbol{P} = \hat{\eta}\boldsymbol{p}$, $\hat{\Omega}$ is a 3×3 matrix with $\hat{\Omega}_{ij} = \omega_i \delta_{ij}$, the constants ω_i, $i = 1, 2, 3$, are the oscillator frequencies along three directions, two constant matrix operators $\hat{\eta}$ and $\hat{\gamma}$ are

$$\hat{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

they satisfy

$$\hat{\eta}^2 = 1, \quad \hat{\gamma}^2 = 1, \quad \{\hat{\gamma}, \hat{\eta}\} = 0. \tag{4}$$

Only $\hat{\zeta}$ is a newly introduced operator.

In (2), the operator $\boldsymbol{P} - im\hat{\gamma}\hat{\Omega} \cdot \boldsymbol{Q}$ is Hermitian and also $\mathcal{PT}$-symmetric. The newly introduced $\mathcal{PT}$-symmetric term $-im\hat{\zeta}\hat{\Omega} \cdot \boldsymbol{Q}$ is not Hermitian because the operator $\hat{\zeta}$ is not anticommuting with $\hat{\eta}$. We choose

$$\hat{\zeta} = \zeta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

where ζ is a real constant.

Through the $\mathcal{PT}$-symmetric minimal substitution (2), Klein-Gordon equation $(c^2\boldsymbol{P}^2 + m^2c^4)\psi = E^2\psi$ becomes

$$c^2[\boldsymbol{p}^2 + (1 - \hat{\zeta}^2)m^2\boldsymbol{q} \cdot \hat{\Omega}^2 \cdot \boldsymbol{q} - i2m\hat{\zeta}\boldsymbol{q} \cdot \hat{\Omega} \cdot \boldsymbol{p} + \hbar m(\hat{\gamma} - \hat{\zeta})Tr\hat{\Omega} + m^2c^2]\psi = E^2\psi. \tag{6}$$

Writing ψ as a two-component wave function

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \tag{7}$$

we obtain the following wave equations

$$\begin{aligned} &c^2[\boldsymbol{p}^2 + (1 - \zeta^2)m^2\boldsymbol{q} \cdot \hat{\Omega}^2 \cdot \boldsymbol{q} - i2m\zeta\boldsymbol{q} \cdot \hat{\Omega} \cdot \boldsymbol{p} - \hbar m(1 + \zeta)Tr\hat{\Omega} + m^2c^2]\psi_a = E_a^2\psi_a, \\ &c^2[\boldsymbol{p}^2 + (1 - \zeta^2)m^2\boldsymbol{q} \cdot \hat{\Omega}^2 \cdot \boldsymbol{q} - i2m\zeta\boldsymbol{q} \cdot \hat{\Omega} \cdot \boldsymbol{p} + \hbar m(1 - \zeta)Tr\hat{\Omega} + m^2c^2]\psi_b = E_b^2\psi_b. \end{aligned} \tag{8}$$

To find exact solutions for this eigenvalue problem we write the ψ_a and ψ_b in (8) in the form

$$\psi_a = \exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\psi_a', \quad \psi_b = \exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\psi_b'. \tag{9}$$

Working out the action of operators $\boldsymbol{p}$ and $\boldsymbol{p}^2$ on wave function, we obtain

$$\begin{aligned}&-i2m\zeta\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{p}\left[\exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\psi'\right]\\ &\quad=\exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)(2\zeta^2m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}-i2m\zeta\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{p})\psi',\end{aligned} \tag{10}$$

$$\begin{aligned}&\boldsymbol{p}^2\left[\exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\psi'\right]\\ &\quad=\exp\left(-\frac{\zeta m \boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)(\hbar m\zeta Tr\hat{\Omega}-\zeta^2m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}+i2m\zeta\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{p}.+\boldsymbol{p}^2)\psi'.\end{aligned} \tag{11}$$

By substituting (10) and (11) into (8) we immediately obtain the following equations

$$\begin{aligned}c^2(\boldsymbol{p}^2+m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}-\hbar mTr\hat{\Omega}+m^2c^2)\psi_a'&=E_a^2\psi_a',\\ c^2(\boldsymbol{p}^2+m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}+\hbar mTr\hat{\Omega}+m^2c^2)\psi_b'&=E_b^2\psi_b',\end{aligned} \tag{12}$$

which is just the Klein-Gordon oscillator wave equations reported by Bruce and Minning [14]. Thus, while the wave equations are different for Hermitian and $\mathcal{PT}$-symmetric Klein-Gordon oscillator, the energy spectrum is exactly the same in both cases. This means that the choice of (5) is correct, since it gives a oscillator energy spectrum which becomes the energy spectrum of a harmonic oscillator in the non-relativistic limit.

An interesting special case of (2) is where $\hat{\gamma}=0$ and $\zeta>0$. In this case, (12) becomes Klein-Gordon equation $(c^2\boldsymbol{P}^2+m^2c^4)\psi'=E^2\psi'$. Therefore, the energy is the same as that of a relativistic free particle. However, the wave function ψ is not plane wave and exponentially vanishing at $\boldsymbol{q}\to\infty$ because of the factor $\exp\{-\zeta m\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}/2\hbar\}$.

Now consider a $\mathcal{PT}$-symmetric Klein-Gordon oscillator moving in a uniform magnetic field $\boldsymbol{B}$, and the magnetic vector potential takes following form:

$$\boldsymbol{A}=\frac{\boldsymbol{B}\times\boldsymbol{Q}}{2}. \tag{13}$$

The magnetic field $\boldsymbol{B}$ is along the z-axis, $\boldsymbol{B}=B\boldsymbol{e}_z$. The Klein-Gordon equation associated with this system can be written as

$$\begin{aligned}&\left[c^2\left(\boldsymbol{P}-\frac{e}{c}\boldsymbol{A}-im\hat{\gamma}\hat{\Omega}\cdot\boldsymbol{Q}-im\hat{\zeta}\hat{\Omega}\cdot\boldsymbol{Q}\right)\right.\\ &\qquad\left.\times\left(\boldsymbol{P}-\frac{e}{c}\boldsymbol{A}-im\hat{\gamma}\hat{\Omega}\cdot\boldsymbol{Q}-im\hat{\zeta}\hat{\Omega}\cdot\boldsymbol{Q}\right)+m^2c^4\right]\psi=E^2\psi.\end{aligned} \tag{14}$$

We will see that, although this wave equation is non-Hermitian and non-$\mathcal{PT}$-symmetric, the eigenvalues are completely real.

Using gauge condition $\nabla\cdot\boldsymbol{A}=0$ and anticommutation relation $\{\hat{\gamma},\hat{\eta}\}=0$, we obtain

$$c^2\bigg[\boldsymbol{p}^2+\frac{e^2}{c^2}\boldsymbol{A}^2+(1-\hat{\zeta}^2)m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}-i2m\hat{\zeta}\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{p}-\frac{e}{c}(\boldsymbol{B}\times\boldsymbol{q})\cdot\boldsymbol{p}$$
$$+i\frac{em\hat{\zeta}}{c}(\boldsymbol{B}\times\boldsymbol{q})\cdot\hat{\Omega}\cdot\boldsymbol{q}+\hbar m(\hat{\gamma}-\hat{\zeta})Tr\hat{\Omega}+m^2c^2\bigg]\psi=E^2\psi. \tag{15}$$

Here, if ψ has the same form as (9) then

$$\left[-\frac{e}{c}(\boldsymbol{B}\times\boldsymbol{q})\cdot\boldsymbol{p}+i\frac{em\zeta}{c}(\boldsymbol{B}\times\boldsymbol{q})\cdot\hat{\Omega}\cdot\boldsymbol{q}\right]\exp\left(-\frac{\zeta m\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\psi'$$
$$=\exp\left(-\frac{\zeta m\boldsymbol{q}\cdot\hat{\Omega}\cdot\boldsymbol{q}}{2\hbar}\right)\left[-\frac{e}{c}(\boldsymbol{B}\times\boldsymbol{q})\cdot\boldsymbol{p}\right]\psi'. \tag{16}$$

We are able therefore to simplify this wave equation, which reduces to the standard Landau problem of Klein-Gordon oscillator [14]

$$c^2\left(\boldsymbol{p}^2+\frac{e^2}{c^2}\boldsymbol{A}^2+m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}-\frac{eB}{c}L_z-\hbar mTr\hat{\Omega}+m^2c^2\right)\psi'_a=E_a^2\psi'_a,$$
$$c^2\left(\boldsymbol{p}^2+\frac{e^2}{c^2}\boldsymbol{A}^2+m^2\boldsymbol{q}\cdot\hat{\Omega}^2\cdot\boldsymbol{q}-\frac{eB}{c}L_z+\hbar mTr\hat{\Omega}+m^2c^2\right)\psi'_b=E_b^2\psi'_b. \tag{17}$$

The definitions of the $\mathcal{PT}$-symmetric Dirac and Duffin-Kemmer-Petiau oscillators [15] are also given in terms of $\mathcal{PT}$-symmetric minimal substitution in a similar manner as the $\mathcal{PT}$-symmetric Klein-Gordon oscillator. Duffin-Kemmer-Petiau equation [26–28] is quite similar to Dirac equation with the change of commutation rules of matrices.

Making the substitution

$$\boldsymbol{p}\to\boldsymbol{p}-i\hat{\beta}m\omega\boldsymbol{r}-i\hat{\zeta}m\omega\boldsymbol{r} \tag{18}$$

in free Dirac equation, we obtain the Dirac equation with a $\mathcal{PT}$-symmetric oscillator interaction

$$[c\hat{\alpha}\cdot(\boldsymbol{p}-i\hat{\beta}m\omega\boldsymbol{r}-i\hat{\zeta}m\omega\boldsymbol{r})+\hat{\beta}mc^2]\psi=E\psi. \tag{19}$$

It is straightforward to see that if ψ is the wave function of a Dirac oscillator, $\exp\{-\zeta m\omega\boldsymbol{r}^2/2\hbar\}\psi$ is the solution of above equation. The energy spectrum of $\mathcal{PT}$-symmetric Dirac oscillator is also the same as that of Hermitian Dirac oscillator.

In conclusion, we have extended Hermitian Klein-Gordon oscillator to a $\mathcal{PT}$-symmetric version. We have discussed the solutions of the eigenvalue equations. We have concluded that energy spectrum of $\mathcal{PT}$-symmetric Klein-Gordon oscillator is the same as that of Hermitian Klein-Gordon oscillator. We have also briefly discussed $\mathcal{PT}$-symmetric Dirac oscillator. It seems interesting to study $\mathcal{PT}$-symmetric oscillators in a noncommutative space. This problem is presently under consideration.

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